Non-singular modified gravity: the unification of the inflation, dark energy and dark mater

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Abstract.
Using the fluid representation, we formulate the conditions for the appearance of all four types finite-time future singularity in modified gravity in accelerating FRW universe. It stressed that number of standard quintessence/phantom DE theories (including scalar, fluid, DBI ones, etc) brings the accelerating cosmology to future singularity precisely in the same way as singular modified gravity DE. The viable non-singular modified gravity unifying the early-time inflation with late-time acceleration is considered. It is shown that adding such non-singular theory to another realistic singular modified gravity which has the accelerating solution with future singularity may cure the singularity of resulting combined model. This universal scenario may be naturally applied to standard singular DE models as well as to inflationary theories with initial singularity. This suggests the additional fundamental reasoning for modification of General Relativity.

INTRODUCTION

Modified gravity suggests the gravitational alternative for the unified description of the early-time inflation with late-time acceleration and dark matter. It is remarkable that in this scenario for the universe evolution there is no need in the introduction of extra dark components like inflaton, quintessence, dark fluid, dark matter particles, etc. The only early-time and late-time gravitational action is changed if compare with General Relativity in such a way that local tests and cosmological bounds are not violated. Hence, the gravitational field is responsible for dark behavior of the universe. The comparison of different modified gravities aimed to the accelerating universe description may be found in ref.[1].

Imagine that we consider the (accelerating) FRW universe. In this case, whatever is modified gravity, it may be represented as General Relativity with some (generalized) fluid. Different properties of modified gravity may be clarified in this representation. For instance, for realistic theories which unify the early-time inflation with late-time acceleration and which we consider here it is clear that they behave as effective phantom/quintessence fluids at the late times. Then, such modified gravities may bring the evolution to the finite-time future singularity of one of four types according to the classification of ref.[2] in the same way as other phantom/quintessence dark energy models. There is no any qualitative difference between modified gravity and other dark energy models in this respect. (Reversing the time flow, the finite-time future singularity is nothing else as initial singularity for number of inflationary theories.)

In the present paper we explicitly demonstrate that the singularity appearance is typical property for number of modified gravity with the effective quintessence/phantom late-time behavior precisely in the same way as for other (scalar/fluid/composite) dark energy models. The appearance of all four types of future singularity in unified modified gravity occurs as it was demonstrated in ref.[3] (for first observation on Big Rip singularity in modified gravity and its avoidance by adding of R^2-term see ref.[4, 5]). The realistic non-singular modified gravity which unifies the early-time inflation with late-time acceleration may be considered as the tool to prevent the future singularity. In particular, it is demonstrated that adding such non-singular theory to another realistic modified gravity which has the accelerating

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solution with future singularity may cure the singularity appearance in combined realistic modified gravity. In the same way, the initial singularity may be cured unless the classical description is valid.

THE FORMULATION OF MODIFIED GRAVITY AS GENERAL RELATIVITY WITH GENERALIZED FLUID AND FINITE-TIME FUTURE SINGULARITIES

Let us start from the general modified gravity with the action:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left(R + f \left(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \Box R, \Box^{-1}R, \cdots \right) \right) + L_{\mathrm{m}} \right\}, \tag{1}$$

where all combinations of local and non-local terms are possible, L_{m} is matter Lagrangian and the function $f(R, \cdots)$ may also contain gravitational partners (say, dilaton, axion, etc. in string-inspired gravity).

We consider spatially-flat FRW universe with scale factor $a(t)$. In all cases for theory (1), it is possible to write the gravitational field equations in the form of standard FRW equations with effective energy-density ρ_{eff} and pressure p_{eff} produced by the extra gravitational terms $F(R, \cdots)$ and L_{m}.

For instance, when $f = f(R)$, one gets

$$\rho_{\mathrm{eff}} = \frac{1}{\kappa^2} \left(-\frac{1}{2}f(R) + 3\left(H^2 + \dot{H}\right)f'(R) - 18\left(4H^2\dot{H} + H\ddot{H}\right)f''(R) \right) + \rho_{\mathrm{matter}}, \tag{2}$$

$$p_{\mathrm{eff}} = \frac{1}{\kappa^2} \left(\frac{1}{2}f(R) - \left(3H^2 + \dot{H}\right)f'(R) + 6\left(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \dddot{H}\right)f''(R) + 36\left(4H\dot{H} + \ddot{H}\right)^2 f'''(R) \right)$$
$$+ p_{\mathrm{matter}}. \tag{3}$$

For details of explicit fluid presentation of $f(R)$ gravity, see [6]. In case of Gauss-Bonnet modified gravity $f(G)$ [7]:

$$\rho_{\mathrm{eff}} = \frac{1}{2\kappa^2} \left[\mathscr{G} f'_{\mathscr{G}}(\mathscr{G}) - f_{\mathscr{G}}(\mathscr{G}) - 24^2 H^4 \left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right) f''_{\mathscr{G}} \right] + \rho_{\mathrm{matter}},$$

$$p_{\mathrm{eff}} = \frac{1}{2\kappa^2} \left[f_{\mathscr{G}}(\mathscr{G}) + 24^2 H^2 \left(3H^4 + 20H^2\dot{H}^2 + 6\dot{H}^3 + 4H^3\ddot{H} + H^2\dddot{H}\right) f''_{\mathscr{G}}(\mathscr{G}) \right.$$
$$\left. - 24^3 H^5 \left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right)^2 f'''_{\mathscr{G}}(\mathscr{G}) \right] + p_{\mathrm{matter}}. \tag{4}$$

In the same way one can get the effective gravitational pressure and energy density so that the equations of motion for arbitrary modified gravity can be rewritten in the universal FRW form typical for General Relativity:

$$\frac{3}{\kappa^2}H^2 = \rho_{\mathrm{eff}}, \quad p_{\mathrm{eff}} = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right). \tag{5}$$

There are just standard FRW gravitational equations.

Formally, the modified gravity equations (5) have just the same form as for GR with matter. The only difference is that the effective energy-density and pressure are caused by extra gravitational terms due to the modification of the GR Lagrangian. It is quite well-known that equations of motion (5) can describe the accelerating (early-time or late-time) epoch when all or part of the (effective) energy conditions are violated. For instance, when $w_{\mathrm{eff}} < -1$, one gets the effective phantom (super)acceleration evolutionary phase (all energy conditions are violated). When $-1 < w_{\mathrm{eff}} < -1/3$ ($\rho_{\mathrm{eff}} = w_{\mathrm{eff}}\rho_{\mathrm{eff}}$), one gets the effective quintessence acceleration epoch, in this case, only part of energy conditions is violated. For $w_{\mathrm{eff}} = -1$, the de Sitter acceleration evolution emerges.

The remarkable property of phantom/quintessence accelerating universe is the occurrence of the finite-time future singularity which is caused by the violation of the energy conditions. The phantom evolution when it is not transient always ends up with so-called Big Rip singularity. The quintessence accelerating evolution may enter (or may not enter) to soft finite-time future singularities. The classification of four possible finite-time future singularities for late-time accelerating universe has being made in ref. [2].

The clarifying remark is in order. Big number of phantom/quintessence dark energy models (say, fluid, scalar, spinor, DBI, tachyon, etc.) brings the evolution to the finite-time future singularity. From the above, equivalent description it is clear that number of modified gravities also describes the effective phantom/quintessence accelerating universe

which evolves to finite-time future singularity. *It is important to understand that the singularity occurrence is general feature of the phantom/quintessence dark energy models which may be fluid, scalar, or modified gravity.* Moreover, even the classification of finite-time singularities for modified gravity is given by the same scheme [2] where p and ρ should be substituted by p_{eff}, ρ_{eff}:

- Type I ("Big Rip") : For $t \to t_s$, $a \to \infty$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$. This also includes the case of ρ_{eff}, p_{eff} being finite at t_s.
- Type II ("sudden") : For $t \to t_s$, $a \to a_s$, $\rho_{\text{eff}} \to \rho_s$ and $|p_{\text{eff}}| \to \infty$
- Type III : For $t \to t_s$, $a \to a_s$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$
- Type IV : For $t \to t_s$, $a \to a_s$, $\rho_{\text{eff}} \to 0$, $|p_{\text{eff}}| \to 0$ and higher derivatives of H diverge. This also includes the case in which p_{eff} (ρ_{eff}) or both of p_{eff} and ρ_{eff} tend to some finite values, while higher derivatives of H diverge.

Indeed, it has been observed the approach to Big Rip singularity in the modified gravity in ref.[4]. However, it was demonstrated there that phantom phase becomes transient due to presence of R^2-term and no singularity occurs. In ref.[5], it was clearly demonstrated the appearance of Big Rip type singularity in several $f(R)$ modified gravity models (see also examples of modified gravity which describe the phantom era and occurrence of Big Rip singularity in [1]).

The explicit construction of $f(R)$ gravity models which lead to all four possible types of future singularity after the corresponding quintessence/phantom era has been presented in ref.[3]. Even more explicit models of $f(R)$ and other modified gravities which realize the four possible finite-time future singularity have been presented in ref.[8]. The search of singular modified gravity models was done using the reconstruction scheme developed in refs.[9]. In this way, the modified gravity which realizes finite-time Type I,II,III or IV future singularity may be constructed in the same way as for traditional DE models (scalar, vector, spinor, etc). Moreover, it was shown in refs.[3, 8, 10], how to make the non-singular model from the initially singular modified gravity. Explicitly it was considered the addition of R^2-term (as in the non-singular modified gravity model [4]), or some other gravitational terms [8] relevant at the early universe, or the account of the quantum contribution due to conformal anomaly as scenario for curing the future singularity of any type. (For related discussion of Type II finite-time future singularity in specific model of $f(R)$ gravity and its avoidance by the same trick [4] of adding R^2-term, see refs.[11]). It is interesting to note that above four types of future singularities in FRW space-time reappear in spherically symmetric space-time as energy density and pressure singularities at finite radius [12]. It is precisely these singularities which may lead to problems in the formation of relativistic stars and black holes in any dark energy model evolving to future singularity. In the next section, we discuss the realistic $f(R)$ gravity which is non-singular and which unifies the early-time inflation with late-time acceleration.

NON-SINGULAR REALISTIC $f(R)$ MODEL UNIFYING THE EARLY-TIME INFLATION WITH LATE-TIME ACCELERATION

Let us consider the following $f(R)$ model [13]:

$$f(R) = \frac{\alpha R^{m+l} - \beta R^n}{1 + \gamma R^l}. \tag{6}$$

For the case $m = l = n$, the model has been analyzed in ref.[13]. It was shown that the model may describe the realistic unification of early-time inflation with late-time acceleration (for a first proposal of the modified gravity with such unification, see [14]). For the realistic unification, $\beta\gamma/\alpha \sim 10^{228(n-1)}\,\text{eV}^2$ in the solar system. This gives extremely small (non-observable) correction to the Newton law if $n \geq 2$. In this case, there is no the observable fifth force (the corresponding force is much below the observational bounds). It is not difficult to check that for $n = 2$, the above model (6) which passes the local tests is also non-singular. Hence, the model (6) suggests the generalization of the scenario to cure the future finite-time singularity via adding the R^2-term [4]. Actually, the same mechanism is used: one can consider the addition of the term $R^2\tilde{f}(R)$ where $\lim_{R\to 0}\tilde{f}(R) = c_1$, $\lim_{R\to\infty}\tilde{f}(R) = c_2$ as the scenario to remove the future singularity. In principle, other higher derivative terms may be also proposed for this purpose. Hence, one can consider the realistic dark energy alternative $f(R)$ gravity from the class suggested in ref.[15] (or more general realistic models [16] unifying the early-time inflation with late-time acceleration).

In case that such theory contains the finite-time future singularity (as it happens in many other dark energy models [2]), one can add R^2-term [4] or $R^2\tilde{f}(R)$ (see above) in order to cure the finite-time singularity. Of course, one can also

take into account quantum effects due to conformal anomaly [2, 8] or quantum gravity effects [17] or generalized fluid of special form [8, 18] to prevent the occurrence of future singularity. Moreover, the use of the function of the form (6) to cure the finite-time singularity seems to be the most promising one. The reason is that the terms of such sort are mainly relevant at the early-time era, so eventually they do not spoil the local tests of the theory. From another side, such terms give the contribution to the cosmological perturbations so probably they may be used to make that more consistent with the observational data.

As the realistic example, we consider the viable theory [16]:

$$f(R) = -\frac{(R-R_0)^{2k+1} + R_0^{2k+1}}{f_0 + f_1 \left\{ (R-R_0)^{2k+1} + R_0^{2k+1} \right\}} . \tag{7}$$

It has been shown [16] that for $k \geq 10$ such modified gravity passes the local tests. It also unifies the early-time inflation with dark energy epoch.

In (7), R_0 is current curvature $R_0 \sim \left(10^{-33} \, \mathrm{eV} \right)^2$. We also require,

$$f_0 \sim \frac{R_0^{2n}}{2} , \quad f_1 = \frac{1}{\Lambda_i} . \tag{8}$$

Here Λ_i is the effective cosmological constant in the inflation epoch. When $R \gg \Lambda_i$, $f(R)$ (7) behaves as

$$f(R) \sim -\frac{1}{f_1} + \frac{f_0}{f_1^2 R^{2n+1}} . \tag{9}$$

We now use the trace equation, which is the trace part of the FRW gravitational equation

$$3\Box f'(R) = R + 2f(R) - Rf'(R) - \kappa^2 T . \tag{10}$$

Here T is the trace of the matter energy-momentum tensor. Then assuming the FRW metric with flat spatial part, one finds

$$R \sim (t_0 - t)^{-2/(2n+3)} , \tag{11}$$

which diverges at finite future time $t = t_0$. By a similar analysis, we can show that if $f(R)$ behaves as $f(R) \sim R^\alpha$ for large R with a constant α, a future singularity appears if $\alpha > 2$ or $\alpha < 0$. Conversely if $2 \geq \alpha \geq 0$, the singularity does not appear. Then by adding the previous term $R^2 \tilde{f}(R)$, where $\lim_{R \to 0} \tilde{f}(R) = c_1$, $\lim_{R \to \infty} \tilde{f}(R) = c_2$, to $f(R)$ in (7), the future singularity (11) disappears.

Let us observe the above situation in more detail. Now we assume

$$f(R) \sim F_0 + F_1 R^\alpha , \tag{12}$$

when R is large. Here F_0 and F_1 are constants where F_0 may vanish but we assume $F_1 \neq 0$. In case of (9), we have

$$F_0 = -\frac{1}{f_1} , \quad F_1 = \frac{f_0}{f_1^2} , \quad \alpha = -(2n+1) . \tag{13}$$

Under the assumption (12), the trace equation (10) gives

$$3F_1 \Box R^{\alpha-1} = \begin{cases} R & \text{when } \alpha < 0 \text{ or } \alpha = 2 \\ (2-\alpha) F_1 R^\alpha & \text{when } \alpha > 1 \text{ or } \alpha \neq 2 \end{cases} . \tag{14}$$

In the FRW background with flat spatial part, when the Hubble rate has a singularity as

$$H \sim \frac{h_0}{(t_0 - t)^\beta} , \tag{15}$$

with constants h_0 and β, the scalar curvature $R = 6\dot{H} + 12H^2$ behaves as

$$R \sim \begin{cases} \frac{12h_0^2}{(t_0-t)^{2\beta}} & \text{when } \beta > 1 \\ \frac{6h_0 + 12h_0^2}{(t_0-t)^2} & \text{when } \beta = 1 \\ \frac{6\beta h_0}{(t_0-t)^{\beta+1}} & \text{when } \beta < 1 \end{cases} . \tag{16}$$

In (15) or (16), $\beta \geq 1$ case corresponds to Type I (Big Rip) singularity, $1 > \beta > 0$ to Type III, $0 > \beta > -1$ to Type II, and $\beta < -1$ but $\beta \neq$ integer to Type IV. By substituting (16) into (14), we find that there are two classes of consistent solutions. The first solution is specified by $\beta = 1$ and $\alpha > 1$ but $\alpha \neq 2$ case, which corresponds to the Big Rip ($h_0 > 0$ and $t < t_0$) or Big Bang ($h_0 < 0$ and $t > t_0$) singularity at $t = t_0$. Another one is $\alpha < 1$, and $\beta = -\alpha/(\alpha - 2)$ ($-1 < \beta < 1$) case, which corresponds to (11) and to the II Type future singularity. In fact, we find $\alpha = -2n - 1$ and therefore $\beta + 1 = -2/(2n + 3)$. We should note that when $\alpha = 2$, that is, $f(R) \sim R^2$, there is no any singular solution. Therefore if we add the above term $R^2 \tilde{f}(R)$, where $\lim_{R \to 0} \tilde{f}(R) = c_1$, $\lim_{R \to \infty} \tilde{f}(R) = c_2$, to $f(R)$ in (7), the added term dominates and modified $f(R)$ behaves as $f(R) \sim R^2$, the future singularity (11) disappears. We also note that if we add R^n-term $n = 3, 4, 5, \cdots$, the singularity becomes (in some sense) worse since this case corresponds to $\alpha = n > 1$, that is Big Rip case. Using the potential which appears when we transform $F(R)$-gravity to scalar-tensor theory [3], it has been found that the future singularity may not appear in case $0 < \alpha < 2$.

Thus, we demonstrated that combination of two realistic modified gravities where one of them is non-singular one may heal the finite-time future singularity of combined realistic model. On the same time, the nice properties of the models: the successful passing of local tests as well as unified description of the inflation with dark energy epoch remain to be the same.

DISCUSSION

In summary, using the fluid representation, we demonstrated that future singularity appearance is typical property of number of modified gravities with the effective quintessence/phantom late-time behavior in the same way as for other simpler (scalar/fluid) dark energy models. These four different future cosmological singularities may manifest themselves as effective pressure/energy-density divergence at finite radius in spherically-symmetric space. This may cause number of problems with black holes and relativistic stars formation for *any singular dark energy* (as is shown in [12]), not only for alternative gravity dark energy models. Moreover, the easiest way to cure the future singularity is again to call for the modification of gravity, for instance, adding the R^2-term which is relevant at the very early universe. Hence, starting from singular fluid/scalar dark energy one is forced to modify its gravitational sector in order to cure the singularity. Is not then more natural to start from the modified gravity from the very beginning? Moreover, the combination of non-singular realistic modified gravity with singular realistic model unifying the early-time inflation with late-time acceleration leads to the resulting realistic non-singular theory. We concentrated mainly on $f(R)$ gravity but other modified gravities may show very similar singular behavior for effective quintessence/phantom late-time era as is explained in ref.[8]. The same universal tools to cure the future singularity may be used for any modified gravity. Finally, it is interesting to mention that above scheme easily accounts for description of DM effects within the same modified gravity (for review of DM properties from modified gravity, see [19]).

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